NONSEMISIMPLE QUANTUM GROUPS AS HOPF ALGEBRAS OF THE DUAL FUNCTIONS

N.A. Gromov
Department of Mathematics, Komi Research Center, Ural Division, Russian Academy of Science, Syktyvkar, 167000, Russia

Abstract. The nonsemisimple quantum Cayley-Klein groups $Fun(SU_q(2;\mathbf{j}))$ are realized as Hopf algebra of the noncommutative functions with the dual (or Study) variables. The *dual* quantum algebras $su_q(2;\mathbf{j})$ are constructed and their isomorphisms with the corresponding quantum orthogonal algebras $so_q(3;\mathbf{j})$ are established. The possible couplings of the Cayley-Klein and Hopf structures are considered.

1 Introduction

The simple (or semisimple) classical Lie groups (algebras) may be transformed by contractions to the nonsemisimple ones of the different structure. We named the set of such groups (algebras) as Cayley-Klein (CK) groups (algebras). In our approach [2] the contractions of the groups (algebras) are described with the help of dual valued parameters, i.e. the nonsemisimple CK groups are regarded as the groups over an associative algebra $D_n(\iota; \mathrm{C})$ with the nilpotent generators $\iota_k, \iota_k^2 = 0, k = 1, ..., n$ satisfying the commutative low of multiplications $\iota_k \iota_m = \iota_m \iota_k \neq 0, k \neq m$. The general element of the dual algebra $D_n(\iota; \mathrm{C})$ have the form

$$a = a_0 + \sum_{p=1}^{2^n-1} \sum_{k_1 < \ldots < k_p} a_{k_1 \ldots k_p} \iota_{k_1} \ldots \iota_{k_p}, \quad a_0, a_{k_1 \ldots k_p} \in \mathrm{C}. \tag{1}$$

For $n = 1$ we have $D_1(\iota_1; \mathrm{C}) \ni a = a_0 + a_1 \iota_1$, i.e. dual (or Study) numbers, when $a_0, a_1 \in \mathrm{R}$. For $n = 2$ the general element of $D_2(\iota_1, \iota_2; \mathrm{C})$ is written as follows: $a = a_0 + a_1 \iota_1 + a_2 \iota_2 + a_{12} \iota_1 \iota_2$.

It is shown in [3], that the representations of $SU(2)$, which are defined in the space of functions on the qroup, are transformed to the representations of the nonsemisimple euclidean qroup $E(2)$, when the complex parameters

of $SU(2)$ are changed by the dual one's, i.e. when the complex variables of the functions are changed by the elements of the dual algebra $D_2(\iota_1, \iota_2; \mathrm{C})$.

In present paper we apply this idea to regard the nonsemisimple CK groups as the groups over the dual algebra $D_n(\iota; \mathrm{C})$ to the case of quantum groups. According with the general theory of quantum groups [1], the quantum CK groups $Fun(SU_q(2;\mathbf{j}))$ are regarded in Sec.2 as Hopf algebra of the noncommutative functions of the dual variables. The *dual*[1] to $Fun(SU_q(2;\mathbf{j}))$ quantum algebras $su_q(2;\mathbf{j})$ are constructed in Sec.3 and their isomorphisms with the corresponding quantum orthogonal CK algebras $so_q(3;\mathbf{j})$ are found in Sec.4. The different dual parametrizations of $Fun(SU_q(2;\mathbf{j}))$ lead to the different *dual* algebras $so_q(3;\mathbf{j})$, which are distinguished by the choise of the primitive element of the Hopf algebra. All such possible combinations of the Hopf and CK structures are considered in Sec.5.

2 Quantum group $Fun(SU_q(2;\mathbf{j}))$

The quantum unitary group $Fun(SU_{\tilde{q}}(2))$ is generated [1] by the matrix with noncommutative elements

$$T = \left(\begin{array}{cc} \tilde{a} & \tilde{b} \\ -\tilde{q}^{-1}\bar{\tilde{b}} & \bar{\tilde{a}} \end{array} \right) = \left(\begin{array}{cc} \tilde{a}_1 + i\tilde{a}_2 & \tilde{b}_1 + i\tilde{b}_2 \\ -e^{-\tilde{z}}(\tilde{b}_1 - i\tilde{b}_2) & \tilde{a}_1 - i\tilde{a}_2 \end{array} \right),$$

$$\det{}_q T = \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{b}_1^2 + \tilde{b}_2^2 = 1, \qquad (2)$$

where the deformation parameter $\tilde{q} = e^{\tilde{z}},\ \tilde{z} \in \mathrm{C}$ and the bar designate the complex conjugation. Commutation relations for generator are defined by the following R-matrix

$$R_{\tilde{z}} = \left(\begin{array}{cccc} \tilde{q} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \tilde{\lambda} & 1 & 0 \\ 0 & 0 & 0 & \tilde{q} \end{array} \right) = \left(\begin{array}{cccc} e^{\tilde{z}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2\sinh\tilde{z} & 1 & 0 \\ 0 & 0 & 0 & e^{\tilde{z}} \end{array} \right), \qquad (3)$$

where $\tilde{\lambda} = \tilde{q} - \tilde{q}^{-1} = 2\sinh\tilde{z}$.

[1]To distinguish the different meanings of the word "dual" we shall write *dual* in the case of the dual aldebra.

Proposition 1. *The quantum group* $Fun(SU_q(2;\mathbf{j}))$, $\mathbf{j}=(j_1,j_2),\ j_k=1,\iota_k,\ k=1,2$ *is given by*

$$T(\mathbf{j})=\left(\begin{array}{cc} a & b \\ -e^{-j_1j_2z}\bar{b} & \bar{a}\end{array}\right)=\left(\begin{array}{cc} a_1+ij_1j_2a_2 & j_1b_1+ij_2b_2 \\ -e^{-j_1j_2z}(j_1b_1-ij_2b_2) & a_1-ij_1j_2a_2\end{array}\right),$$

$$\det{}_qT(\mathbf{j})=a_1^2+j_1^2j_2^2a_2^2+j_1^2b_1^2+j_2^2b_2^2=1, \qquad (4)$$

$$R_z(\mathbf{j})T_1(\mathbf{j})T_2(\mathbf{j})=T_2(\mathbf{j})T_1(\mathbf{j})R_z(\mathbf{j}),$$

$$T_1(\mathbf{j})=T(\mathbf{j})\otimes I, \quad T_2(\mathbf{j})=I\otimes T(\mathbf{j}), \qquad (5)$$

$$\triangle T(\mathbf{j})=T(\mathbf{j})\dot{\otimes}T(\mathbf{j}), \quad \epsilon(T(\mathbf{j}))=I, \qquad (6)$$

$$S(T(\mathbf{j}))=\left(\begin{array}{cc} a_1-ij_1j_2a_2 & -e^{j_1j_2z}(j_1b_1+ij_2b_2) \\ j_1b_1-ij_2b_2 & a_1+ij_1j_2a_2\end{array}\right), \qquad (7)$$

where

$$R_z(\mathbf{j})=\left(\begin{array}{cccc} e^{j_1j_2z} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2\sinh j_1j_2z & 1 & 0 \\ 0 & 0 & 0 & e^{j_1j_2z}\end{array}\right). \qquad (8)$$

In explicit form the commutation relation are as follows

$$[b_1,b_2]=0, \quad [a_1,a_2]=-i(j_1^2b_1^2+j_2^2b_2^2)e^{j_1j_2z}j_1^{-1}j_2^{-1}\sinh j_1j_2z,$$

$$\begin{array}{rcl} a_1b_1 & = & b_1a_1\cosh j_1j_2z+ij_1j_2b_1a_2\sinh j_1j_2z, \\ a_1b_2 & = & b_2a_1\cosh j_1j_2z+ij_1j_2b_2a_2\sinh j_1j_2z, \\ a_2b_1 & = & b_1a_2\cosh j_1j_2z-ib_1a_1j_1^{-1}j_2^{-1}\sinh j_1j_2z, \\ a_2b_2 & = & b_2a_2\cosh j_1j_2z-ib_2a_1j_1^{-1}j_2^{-1}\sinh j_1j_2z. \end{array} \qquad (9)$$

In particular for the most contracted case $j_1=\iota_1, j_2=\iota_2$ we have from Eq.(4) $a_1=1$ and Eqs.(9) become as follows:

$$[b_1,b_2]=0,\ [b_1,a_2]=izb_1,\ [b_2,a_2]=izb_2. \qquad (10)$$

The matrix (4) and R-matrix (8) are obtained from (2) and (3) by the following transformations of generators and deformation parameter:

$$\tilde{z}=j_1j_2z,\ \tilde{a}_1=a_1,\ \tilde{a}_2=j_1j_2a_2,\ \tilde{b}_1=j_1b_1,\ \tilde{b}_2=j_2b_2. \qquad (11)$$

3 $su_q(2;\mathbf{j})$ as the dual to $Fun(SU_q(2;\mathbf{j}))$

The generators of the *dual* to $Fun(SU_{\tilde{q}}(2))$ quantum algebra $su_{\tilde{q}}(2)$ are written [1] in compact matrix form as

$$L^{(+)} = \left(\begin{array}{cc} \tilde{t} & \tilde{u}_1 + i\tilde{u}_2 \\ 0 & \tilde{t}^{-1} \end{array} \right), \qquad L^{(-)} = \left(\begin{array}{cc} \tilde{t}^{-1} & 0 \\ -e^{\tilde{z}}(\tilde{u}_1 - i\tilde{u}_2) & \tilde{t} \end{array} \right). \qquad (12)$$

Following [1], we define the quantum algebra $su_q(2;\mathbf{j})$, *dual* to $Fun(SU_q(2;\mathbf{j}))$ as

$$< L^{(\pm)}(\mathbf{j}), T(\mathbf{j}) >= R^{(\pm)}(\mathbf{j}), \qquad (13)$$

where $L^{(\pm)}(\mathbf{j})$ are given by

$$L^{(+)}(\mathbf{j}) = \left(\begin{array}{cc} t & u \\ 0 & t^{-1} \end{array} \right) = \left(\begin{array}{cc} t & j_1^{-1}u_1 + ij_2^{-1}u_2 \\ 0 & t^{-1} \end{array} \right),$$

$$L^{(-)}(\mathbf{j}) = \left(\begin{array}{cc} t^{-1} & 0 \\ -e^{j_1j_2z}\bar{u} & t \end{array} \right) = \left(\begin{array}{cc} t^{-1} & 0 \\ -e^{j_1j_2z}(j_1^{-1}u_1 - ij_2^{-1}u_2) & t \end{array} \right), \qquad (14)$$

and act on the first order polinomial of the generators of $Fun(SU_q(2;\mathbf{j}))$. $R^{(\pm)}(\mathbf{j})$ are expressed by R-matrix (8) as

$$\begin{array}{l} R(\mathbf{j}) = e^{-j_1j_2z}R_z(\mathbf{j}), \qquad \det R(\mathbf{j}) = 1, \\ R^{(-)}(\mathbf{j}) = R^{-1}(\mathbf{j}), \quad R^{(+)}(\mathbf{j}) = PR(\mathbf{j})P, \\ P(a \otimes b) = b \otimes a. \end{array} \qquad (15)$$

The matrices (14) may be obtained from (12) by the following (contraction) transformations of the generators and the deformation parameter

$$\tilde{z} = j_1j_2z, \quad \tilde{u_1} = u_1/j_1, \quad \tilde{u}_2 = u_2/j_2, \quad \tilde{H} = H/j_1j_2, \qquad (16)$$

where $\tilde{t} = \exp(\tilde{z}\tilde{H}/2), t = \exp(zH/2)$. In explicit form the actions (13) are given by

$$\begin{array}{c} t(b) = t(\bar{b}) = u(b) = \bar{u}(\bar{b}) = u(a) = u(\bar{a}) = 0, \\ t(a) = x, \quad t(\bar{a}) = x^{-1}, \quad u(\bar{b}) = -x\lambda, \quad \bar{u}(b) = x^{-1}\lambda, \end{array} \qquad (17)$$

where $x = e^{j_1j_2z/2}, \ \lambda = 2\sinh j_1j_2z$. Equations (17) become

$$u_k(a_1) = u_k(a_2) = 0, \ t(a_1) = \cosh j_1j_2z/2, \ t(a_2) = -ij_1^{-1}j_2^{-1}\sinh j_1j_2z/2,$$

$$
\begin{aligned}
u_k(b_k) &= -\sinh j_1 j_2 z \cdot \sinh j_1 j_2 z/2, \; k=1,2, \\
u_1(b_2) &= -ij_1^2 \frac{\sinh j_1 j_2 z}{j_1 j_2} \cdot \cosh j_1 j_2 z/2, \\
u_2(b_1) &= ij_2^2 \frac{\sinh j_1 j_2 z}{j_1 j_2} \cdot \cosh j_1 j_2 z/2. \qquad (18)
\end{aligned}
$$

Remark 1. Eqs.(18) for $j_1 = j_2 = 1$ describe the quantum algebra $su_q(2)$.

Remark 2. At first sight nondiagonal elements of $L^{(\pm)}(\mathbf{j})$ (14) are not defined for dual values of j_k, since the division of a real or complex number by the dual units ι_k are not defined. But the matrices $L^{(\pm)}(\mathbf{j})$ are a *linear* functionals on the dual variables $j_k b_k$, therefore their *actions* on the generators of $Fun(SU_q(2;\mathbf{j}))$ gives the well defined expressions (18).

Proposition 2. *The quantum algebra $su_q(2;\mathbf{j})$ is given by*

$$
R^{(+)}(\mathbf{j}) L_1^{(+)}(\mathbf{j}) L_2^{(-)}(\mathbf{j}) = L_2^{(-)}(\mathbf{j}) L_1^{(+)}(\mathbf{j}) R^{(+)}(\mathbf{j}), \qquad (19)
$$

$$
\begin{aligned}
u_1 t &= t u_1 \cosh j_1 j_2 z + ij_1^2 t u_2 \frac{\sinh j_1 j_2 z}{j_1 j_2}, \\
u_2 t &= t u_2 \cosh j_1 j_2 z - ij_2^2 t u_1 \frac{\sinh j_1 j_2 z}{j_1 j_2},
\end{aligned}
$$

$$
[j_1^{-1} u_1, j_2^{-1} u_2] = -2ie^{-j_1 j_2 z} \cdot \sinh j_1 j_2 z \cdot \sinh zH. \qquad (20)
$$

$$
\begin{gathered}
\triangle L^{(\pm)}(\mathbf{j}) = L^{(\pm)}(\mathbf{j}) \dot{\otimes} L^{(\pm)}(\mathbf{j}), \quad \epsilon(L^{(\pm)}(\mathbf{j})) = I, \\
\triangle t = t \otimes t, \quad \triangle u_k = t \otimes u_k + u_k \otimes t^{-1}, \quad k=1,2, \\
\epsilon(t) = 1, \quad \epsilon(u_1) = \epsilon(u_2) = 0, \qquad (21)
\end{gathered}
$$

$$
\begin{gathered}
S(L^{(+)}(\mathbf{j})) = \begin{pmatrix} t^{-1} & -e^{j_1 j_2 z}(j_1^{-1} u_1 + ij_2^{-1} u_2) \\ 0 & t \end{pmatrix}, \\
S(L^{(-)}(\mathbf{j})) = \begin{pmatrix} t & 0 \\ (j_1^{-1} u_1 - ij_2^{-1} u_2) & t^{-1} \end{pmatrix}. \qquad (22)
\end{gathered}
$$

4 Isomorphism of $su_q(2;\mathbf{j})$ and $so_q(3;\mathbf{j})$

We have described the quantum algebra $su_q(2;\mathbf{j})$ as the *dual* to $Fun(SU_q(2;\mathbf{j}))$. In this section we will show their isomorphism with the orthogonal CK algebra $so_q(3;\mathbf{j})$. The quantum analogue of the universal enveloping algebra of $so(3;\mathbf{j})$ with the rotation generator X_{02} is the primitive element has been given in [4]:

Proposition 3. *The Hopf algebra structure of $so_q(3;\mathbf{j};X_{02})$ is given by*

$$\triangle X_{02} = I \otimes X_{02} + X_{02} \otimes I,$$

$$\triangle X = e^{-\hat{z}X_{02}/2} \otimes X + X \otimes e^{\hat{z}X_{02}/2}, \quad X = X_{01}, X_{12},$$

$$\epsilon(X_{01}) = \epsilon(X_{02}) = \epsilon(X_{12}) = 0, \quad \tilde{S}(X_{02}) = -X_{02},$$

$$\tilde{S}(X_{01}) = -X_{01}\cos j_1j_2\hat{z}/2 + j_1^2 X_{12}\frac{\sin j_1j_2\hat{z}/2}{j_1j_2},$$

$$\tilde{S}(X_{12}) = -X_{12}\cos j_1j_2\hat{z}/2 - j_2^2 X_{01}\frac{\sin j_1j_2\hat{z}/2}{j_1j_2},$$

$$[X_{01}, X_{02}] = j_1^2 X_{12}, \quad [X_{02}, X_{12}] = j_2^2 X_{01}, \quad [X_{12}, X_{01}] = \frac{\sinh(\hat{z}X_{02})}{\hat{z}}. \tag{23}$$

Proposition 4. *The Hopf algebra $su_q(2;\mathbf{j})$ (19)–(22) is isomorphic to the Hopf algebra $so_q(3;\mathbf{j};X_{02})$.*

Proof. It is easily verify, that the equations of the previous section are passed in the corresponding expression (23) by the following substitution of the deformation parameter z and the generators $t, j_k^{-1}u_k$ for the deformation parameter $\hat{z}$ and the generators X:

$$z = i\hat{z}/2, \quad t = e^{zH/2}, \quad H = -2iX_{02},$$

$$j_1^{-1}u_1 = 2ij_1De^{-ij_1j_2\hat{z}/4}X_{12}, \quad j_2^{-1}u_2 = 2ij_2De^{-ij_1j_2\hat{z}/4}X_{01},$$

$$D = i\left(\frac{\hat{z}}{2j_1j_2}\sin(j_1j_2\hat{z}/2)\right)^{1/2}. \tag{24}$$

The different definitions of the antipode $\tilde{S}$ in [4] as

$$\tilde{S}(X) = -e^{\hat{z}X_{02}/2}Xe^{-\hat{z}X_{02}/2} = -e^{zH/2}Xe^{-zH/2}, \tag{25}$$

and the antipode (22) of $su_q(2;\mathbf{j})$ as

$$S(u_k) = -e^{-zH/2} u_k e^{zH/2}, \qquad (26)$$

must be taken into account. This produce a slight difference of signs when Eqs. (22) are transformed by (24) as compared with Eqs. (23).

5 On the possible couplings of Hopf and - Cayley–Klein structures

The quantum CK algebra $so_q(n+1;\mathbf{j})$ can be got starting from the quantum orthogonal algebra $so_q(n+1)$ by the following (contraction) transformations of the generators and deformation parameter

$$X_{\mu\nu} = J_{\mu\nu}\tilde{X}_{\mu\nu}, \quad J_{\mu\nu} = \prod_{l=\mu+1}^{\nu} j_l, \quad \mu < \nu, \quad \tilde{z} = Jz, \qquad (27)$$

where the explicit expression for multiplier J depends on the choose of the primitive elements of the Hopf algebra. Note that for nonquantum case the transformations of generators (27) give the CK algebra $so(n+1;\mathbf{j})$ [2], [5], [6].

Under introduction of a Hopf algebra structure in the universal enveloping algebra some commuting generators of $so_q(n+1)$ (the basic ones in the Cartan subalgebra) have to be selected as the primitive elements of the Hopf algebra, i.e. they are distinguished among other generators. Let us observe, that any permutation $\sigma \in S(n+1)$ indices of the rotation generators $\tilde{X}_{\mu\nu}$ of $so_q(n+1)$ (i.e. the transformations from the Weyl group) leads to the isomorphic Hopf algebra $so_q^{'}(n+1)$, but with some other set of primitive generators. This isomorphism of the Hopf algebras may be destroyed by contractions. Indeed, under transformations (27) the primitive generators of $so_q(n+1)$ and $so_q^{'}(n+1)$ are multiplied by the different $J_{\mu\nu}$ and for the specific dual values of j_k, this leads to the nonisomorphic quantum algebras $so_q(n+1;\mathbf{j})$ and $so_q^{'}(n+1;\mathbf{j})$. In other words, the Hopf and CK structures may be combined in a different manner for quantum CK algebras.

Let us illustrate the different couplings of Hopf and CK structures for the case of $so_q(3;\mathbf{j})$. The Hopf algebra $so_q(3;\tilde{X}_{02})$ with $\tilde{X}_{02}$ as the primitive

element is described by Eqs. (23) for $j_1 = j_2 = 1$. Transformations (27) are in this case as follows:

$$X_{01} = j_1 \tilde{X}_{01}, \quad X_{02} = j_1 j_2 \tilde{X}_{02}, \quad X_{12} = j_2 \tilde{X}_{12}, \quad \tilde{\hat{z}} = j_1 j_2 \hat{z} \tag{28}$$

and give in result Eqs. (23). The transformation law of the deformation parameter is defined by those of the primitive generator $\tilde{X}_{02}$, i.e. $J = j_1 j_2$. The substitution $\sigma \in S(3),\ \sigma(0) = 1, \sigma(1) = 0, \sigma(2) = 2$ indices of the generators $\tilde{X}_{\mu\nu}$ leads to the Hopf algebra $so_q(3; \tilde{X}_{12})$ with $\tilde{X}_{12}$ as the primitive element. The introduction of the CK structure by the transformations of generators as in Eqs.(28) and the deformation parameter like $\tilde{X}_{12}$, i.e. $\tilde{\hat{z}} = j_2 \hat{z}$ leads to the following proposition.

Proposition 5. *The quantum algebra* $so_q(3; \mathbf{j}; X_{12})$ *is given by*

$$\triangle X_{12} = I \otimes X_{12} + X_{12} \otimes I,$$

$$\triangle X = e^{-X_{12}\hat{z}/2} \otimes X + X \otimes e^{X_{12}\hat{z}/2}, \quad X = X_{01}, X_{02},$$
$$\epsilon(X_{01}) = \epsilon(X_{02}) = \epsilon(X_{12}) = 0, \quad \tilde{S}(X_{12}) = -X_{12},$$
$$\tilde{S}(X_{01}) = -X_{01} \cos j_2 \hat{z}/2 - X_{02} j_2^{-1} \sin j_2 \hat{z}/2,$$
$$\tilde{S}(X_{02}) = -X_{02} \cos j_2 \hat{z}/2 + X_{01} j_2 \sin j_2 \hat{z}/2,$$
$$[X_{01}, X_{02}] = j_1^2 \hat{z}^{-1} \sinh(\hat{z} X_{12}),\ [X_{02}, X_{12}] = j_2^2 X_{01},\ [X_{12}, X_{01}] = X_{02} \tag{29}$$

Proposition 6. *The quantum qroup* $Fun(SU_q(2; \mathbf{j}))$ *dual to* (29) *is given by relations* (5),(6) *with*

$$T(\mathbf{j}) = \begin{pmatrix} a_1 + i j_2 a_2 & j_1(b_1 + i j_2 b_2) \\ -e^{-j_2 z} j_1 (b_1 - i j_2 b_2) & a_1 - i j_2 a_2 \end{pmatrix},$$
$$\det{}_q T(\mathbf{j}) = a_1^2 + j_2^2 a_2^2 + j_1^2 b_1^2 + j_1^2 j_2^2 b_2^2 = 1, \tag{30}$$

and

$$R_z(\mathbf{j}) = \begin{pmatrix} e^{j_2 z} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2\sinh j_2 z & 1 & 0 \\ 0 & 0 & 0 & e^{j_2 z} \end{pmatrix}. \tag{31}$$

The matrices (30),(31) are obtained from (2),(3) by transformations

$$\tilde{z} = j_2 z,\ \tilde{a}_1 = a_1,\ \tilde{a}_2 = j_2 a_2,\ \tilde{b_1} = j_1 b_1,\ \tilde{b_2} = j_1 j_2 b_2. \tag{32}$$

The *dual* to (30) quantum algebra is defined by Eqs.(13), where the matrices $L^{(\pm)}(\mathbf{j})$ are as follows

$$L^{(+)}(\mathbf{j}) = \begin{pmatrix} t & j_1^{-1}(u_1 + i j_2^{-1} u_2) \\ 0 & t^{-1} \end{pmatrix},$$

$$L^{(-)}(\mathbf{j}) = \begin{pmatrix} t^{-1} & 0 \\ -e^{j_2 z} j_1^{-1}(u_1 - i j_2^{-1} u_2) & t \end{pmatrix} \tag{33}$$

and are obtained from (12) by (contraction) transformations

$$\tilde{z} = j_2 z, \quad \tilde{H} = H/j_2, \quad \tilde{u}_1 = u_1/j_1, \quad \tilde{u}_2 = u_2/j_1 j_2, \tag{34}$$

where $t = \exp(zH/2)$. Nonzero actions of t, u_k on a, b_k are found from the relations (13) in the form

$$t(a_1) = \cosh j_2 z/2, \quad t(a_2) = -i j_2^{-1} \sinh j_2 z/2,$$

$$u_1(b_1) = u_2(b_2) = -\sinh j_2 z \cdot \sinh j_2 z/2,$$
$$u_1(b_2) = -i j_2^{-1} \sinh j_2 z \cdot \cosh j_2 z/2,$$
$$u_2(b_1) = i j_2 \sinh j_2 z \cdot \cosh j_2 z/2, \tag{35}$$

and commutators follow from Eqs.(19)

$$[H, u_1] = -2i u_2, \quad [H, u_2] = 2i j_2^2 u_1,$$

$$[u_2, u_1] = 2i j_1^2 j_2 e^{-j_2 z} \sinh j_2 z \cdot \sinh zH. \tag{36}$$

Proposition 7. *Hopf algebra* (33)–(36) *is isomorphic to* $so_q(3; \mathbf{j}; X_{12})$ (29).

Proof. The connections of H, u_k with the rotation generators $X_{\mu\nu}$ of $so_q(3; \mathbf{j}; X_{12})$ are given by

$$z = i\hat{z}/2, \quad H = -2i X_{12}, \quad u_1 = F \cdot X_{02}, \quad j_2^{-1} u_2 = -j_2 F \cdot X_{01},$$

$$F = e^{-i j_2 \hat{z}/4} \left(2\hat{z} j_2^{-1} \sin(j_2 \hat{z}/2)\right)^{1/2}. \tag{37}$$

As it was mention early the contractions of quantum group and algebras are correspond to the dual values of the parameters j_k. In particular, the quantum euclidean algebra $so_q(3;\iota_1,1;X_{12})$ is described by (29) for $j_1 = \iota_1, j_2 = 1$. The deformation parameter is left untoched, since $j_2 = 1$. Their *dual* euclidean quantum group is realized according with (30) as Hopf algebra of the noncommutative functions with dual variables (cf. [7]–[10]).

The third possible coupling of Hopf and CK structures is connected with the choose of $\tilde{X}_{01}$ as the primitive element and leads to the quantum algebra $so_q(3;\tilde{X}_{01})$.

Proposition 8. *The quantum CK algebra* $so_q(3;\mathbf{j};X_{01})$ *is given by*

$$\triangle X_{01} = I \otimes X_{01} + X_{01} \otimes I,$$

$$\triangle X = e^{-X_{01}\hat{z}/2} \otimes X + X \otimes e^{X_{01}\hat{z}/2}, \quad X = X_{02}, X_{12},$$

$$\epsilon(X_{01}) = \epsilon(X_{02}) = \epsilon(X_{12}) = 0, \quad \tilde{S}(X_{01}) = -X_{01},$$

$$\tilde{S}(X_{02}) = -X_{02}\cos j_1\hat{z}/2 - X_{12}j_1 \sin j_1\hat{z}/2,$$

$$\tilde{S}(X_{12}) = -X_{12}\cos j_1\hat{z}/2 + X_{02}j_1^{-1} \sin j_1\hat{z}/2,$$

$$[X_{01},X_{02}] = j_1^2 X_{12},\ [X_{12},X_{01}] = X_{02},\ [X_{02},X_{12}] = j_2^2 \frac{\sinh \hat{z}X_{01}}{\hat{z}}. \qquad (38)$$

Last equations are obtained in a three steps: i) put $j_1 = j_2 = 1$ in Eqs. (23), ii) apply permutation $\sigma \in S(3)$, $\sigma(0) = 0$, $\sigma(1) = 2$, $\sigma(2) = 1$, iii) introduce CK structure by transformation (28) of generators and deformation parameter like $\tilde{X}_{01}$, i.e. $\tilde{\hat{z}} = j_1\hat{z}$. The *dual* to $so_q(3;\mathbf{j};X_{01})$ quantum group $Fun(SU_q(2;\mathbf{j}))$ is obtained from (2) by the transformations

$$\tilde{z} = j_1 z,\ \tilde{a}_1 = a_1,\ \tilde{a}_2 = j_1 a_2,\ \tilde{b}_1 = j_1 j_2 b_1,\ \tilde{b}_2 = j_2 b_2, \qquad (39)$$

and the matrices $L^{(\pm)}(\mathbf{j})$ are obtained from (12) by transformations

$$\tilde{z} = j_1 z, \quad \tilde{H} = H/j_1, \quad \tilde{u}_1 = u_1/j_1 j_2, \quad \tilde{u}_2 = u_2/j_2. \qquad (40)$$

Eqs. (38) for $j_1 = \iota_1, j_2 = 1$ describe the quantum euclidean algebra, which has been obtained in [8] by contraction of $su_q(2)$. The deformation parameter is transformed in this case.

Eqs. (23) and (29) for $j_1 = \iota_1, j_2 = \iota_2$ give two quantum galilean algebras, which has been obtained in [4], [8]. (Galilean algebra (38) is isomorphic to the algebra (29) for $j_1 = \iota_1, j_2 = \iota_2$).

6 Conclusion

Contractions are the method of receiving a new Lie groups (algebras) from the unitial ones, in particular, the nonsemisimple qroups (algebras) from the simple ones. In the traditional approach [11] this is achieved by introduction of a real zero tending parameter $\epsilon \to 0$. In our approach [2], [5], [6] contractions are described by the dual valued parameters j_k. In the case of standart matrix theory of quantum groups these contractions supplemented with the appropriate transformations of the deformation parameter lead to realization of nonsemisimple quantum qroups as Hopf algebras of noncommutative functions with dual variables. Although both descriptions of contractions are equivalent in many respects, it seems that the language of dual (or Study) numbers is mathematically more correct. For example, the quantum euclidean qroup is described by the matrix (30) for $j_1 = \iota_1, j_2 = 1$ with dual nondiagonal elements, while the limit $l \to 0$ in the traditional approach gives the matrix with zero nondiagonal elements (cf. (16) in [9]). The dual units are nilpotent like the Grassmanian ones, which are used for description of supersymmetry. The only difference is the commutative or anticommutative law of multiplications. The dual algebra $D_n(\iota; \mathrm{C})$ is the subalgebra of the even part of the Grassmanian algebra with $2n$ nilpotent generators.

The constructive algorithm for the description of a different couplings of Hopf and CK structures is given. The different combinations of Hopf and CK structures for the quantum algebras $so_q(3; \mathbf{j})$ are appear on the level of the quantum groups $Fun(SU_q(2; \mathbf{j}))$ as the different dual functions for the elements of matrix T (cf.(4),(30)). The transformation law of the deformation parameter is the same as the one of the dual part of the diagonal elements of T (cf.(11),(32),(40)).

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